Two-dimensional flow in a porous medium with general anisotropy

P.A. Tyvand & A.R.F. Storhaug
Department of Mathematical Sciences and Technology
Norwegian University of Life Sciences
1432 Ås
Norway

peder.tyvand@umb.no
Darcy’s law for flow in an isotropic porous medium

\[ 0 = -\nabla p - \frac{\mu}{K} \vec{v} \]

\( p \) is pressure
\( \mu \) is dynamic viscosity
\( K \) is permeability
\( \vec{v} \) is velocity (vector)

Henry Darcy (1956) formulated this linear law where a driving force is balanced by a resistance. Analogous linear constitutive laws are:

- Ohm’s law of electric conduction
- Fourier’s law of heat conduction
- Fick’s law of mass diffusion

Essentially these are first-order Taylor expansions
Darcy’s law for flow in an isotropic porous medium: Newton’s 2. law

\[ 0 = -\nabla p - \frac{\mu}{K} \mathbf{v} \]

mass x acceleration = pressure force + resistance force (taken per volume, with acceleration term neglected)

Basic experiment for obtaining Darcy’s law
Basic 1D experiment for obtaining Darcy’s law

Tube (a)-(b) filled with a porous medium. Pressure fall \( \Delta p = p_a - p_b \). Tube length \( L \). Cross-section area \( A \). Volume flux \( Q \).

\[
Q = K \frac{A}{\mu} \frac{p_a - p_b}{L} = -K \frac{A}{\mu} \frac{\partial p}{\partial x}
\]

This is the observed relationship between pressure fall and volume flux. The permeability \( K \) is introduced as a proportionality constant.
Darcy’s law for flow in an isotropic porous medium

\[ 0 = -\nabla p - \frac{\mu}{K} \vec{v} \]

\( p \) is pressure
\( \mu \) is dynamic viscosity
\( K \) is permeability
\( \vec{v} \) is velocity (vector)

Follows by generalizing to 3D the empirical relationship

\[ Q = K \frac{A}{\mu} \frac{p_a - p_b}{L} = -K \frac{A}{\mu} \frac{\partial p}{\partial x} \]

by introducing as velocity the specific flux \( v_x = Q/A \). Alternatively one may work with the average pore velocity \( nQ/A \) (\( n \) is porosity).
Darcy’s law for 1D flow in an anisotropic porous medium

The 1D version of Darcy’s law

\[ Q = K \frac{A}{\mu L} \left( p_a - p_b \right) = -K \frac{A}{\mu} \frac{\partial p}{\partial x} \]

is rewritten for an anisotropic medium as

\[ v_x = \frac{Q_x}{A} = -\frac{K_{xx}}{\mu} \frac{\partial p}{\partial x} \]

where \( K_{xx} \) is the permeability for flow in the \( x \) direction as driven by a gradient in the \( x \) direction.
Darcy’s law for 1D flow in an anisotropic porous medium

Darcy’s law for an anisotropic porous medium with 1D pressure gradient $\partial p/\partial x=\Delta p/L_x$

\[
\begin{align*}
v_x &= \frac{Q_x}{A_x} = -\frac{K_{xx}}{\mu} \frac{\partial p}{\partial x} \\
v_y &= \frac{Q_y}{A_y} = -\frac{K_{yx}}{\mu} \frac{\partial p}{\partial x} \\
v_z &= \frac{Q_z}{A_z} = -\frac{K_{zx}}{\mu} \frac{\partial p}{\partial x}
\end{align*}
\]

$K_{yx}, K_{zx}$ are the permeabilities for flows in the $y$ and $z$ directions as driven by a gradient in the $x$ direction. Given by $K_{yx}/K_{xx}=v_y/v_x$, $K_{zx}/K_{xx}=v_z/v_x$
Darcy’s law for 3D flow in an anisotropic porous medium

Darcy’s law for an anisotropic porous medium with 3D pressure gradient \((\partial p/\partial x, \partial p/\partial y, \partial p/\partial z)\)

\[
\mu \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = - \begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix} \begin{pmatrix} \partial p / \partial x \\ \partial p / \partial y \\ \partial p / \partial z \end{pmatrix}
\]

or, in index form

\[
\mu v_i = K_{ij} \frac{\partial p}{\partial x_j}
\]

\(K_{ij} (i=1,2,3) (j=1,2,3)\) are the permeabilities for flow in the \(x_1=x, x_2=y\) and \(x_3=z\) directions as driven by gradients in the \(x_j (j=1,2,3)\) directions
3D anisotropic porous media

- We consider a homogeneous porous medium with general 3D anisotropy
- The anisotropy can be caused by fibres with preferred directions
- The anisotropy can be caused by periodic layering
- These cases may have transverse isotropy: permeability along across layer. permeability in perpendicular direction
- Our analysis will assume general anisotropy, with arbitrary directions of principal axes
Flow in a 3D anisotropic porous medium

Coordinate axes $x, y, z$. Principal axes of permeability $X, Y, Z$, with permeabilities $K_I, K_{II}, K_{III}$. The two coordinate systems are linked by the three Euler angles $\alpha, \beta, \gamma$. 
Flow in a 3D anisotropic porous medium

Principal axes of permeability are $X$, $Y$, $Z$. The permeability matrix is

$$
\begin{pmatrix}
K_I & 0 & 0 \\
0 & K_{II} & 0 \\
0 & 0 & K_{III}
\end{pmatrix}
$$

in the $X$, $Y$, $Z$ system. The matrix is transformed to the $x,y,z$ system.
2D flow in a 3D anisotropic porous medium

The permeability matrix in the $x,y,z$ system

$$
\begin{pmatrix}
K_{11} & K_{12} & K_{13} \\
K_{21} & K_{22} & K_{23} \\
K_{31} & K_{32} & K_{33}
\end{pmatrix}
$$

is given by the three principal permeabilities and the Euler angles (formulas omitted)

Now we restrict our attention to general 2D flow ($\partial/\partial z = 0$)
2D flow in a 3D anisotropic porous medium

- We introduce the pressure gradient that drives a 2D flow \((G_x, G_y) = - (\partial p/\partial x, \partial p/\partial y)\)
- Note that this flow has a passive component in the \(z\) direction, even though there is no pressure gradient in the \(z\) direction.
- Darcy’s law for the 3D porous medium is written as:
  \[
  \vec{v} = K \cdot \vec{G} = \vec{G} \cdot K
  \]
- Here \(K\) is the 3D permeability tensor (matrix), which is symmetric. The velocity vector \(\vec{v}\) has three components \((v_x, v_y, v_z)\)

The idea is now to introduce an effective 2D description for the flow in the \((x,y)\) plane. \(K_1\) and \(K_2\) are defined as effective 2D principal permeabilities, with axes rotated an angle \(\phi\) compared with the coordinate axes \(x\) and \(y\).
2D flow in a 3D anisotropic porous medium

- Darcy's law for 2D flow in a 3D porous medium with \((G_x, G_y) = (-\frac{\partial p}{\partial x}, -\frac{\partial p}{\partial y})\)

- \[ \vec{v} = K \cdot \vec{G} = \vec{G} \cdot K \]

- The velocity vector \(v\) has three components \((v_x, v_y, v_z)\), but we are only interested in its 2D projection \((v_x, v_y)\), which is given by the 2D formula

\[
(v_x, v_y) = \vec{G} \cdot K_{\text{eff}}
\]

An effective 2D description for the flow in the \((x,y)\) plane takes \(K_1\) and \(K_2\) as effective 2D principal permeabilities, with axes rotated an angle \(\phi\)

\[
K_{\text{eff}} = \begin{pmatrix}
K_1 \cos^2 \phi + K_2 \sin^2 \phi & (1/2)(K_1 - K_2) \sin 2\phi \\
(1/2)(K_1 - K_2) \sin 2\phi & K_1 \sin^2 \phi + K_2 \cos^2 \phi
\end{pmatrix}
\]
2D flow in a 3D anisotropic porous medium

- The equations governing $K_1$, $K_2$ and $\phi$ are shown here. They can easily be solved numerically.

\[
K_1 \cos^2 \phi + K_2 \sin^2 \phi = K_I a_{11}^2 + K_{II} a_{21}^2 + K_{III} a_{31}^2 \\
K_1 \sin^2 \phi + K_2 \cos^2 \phi = K_I a_{12}^2 + K_{II} a_{22}^2 + K_{III} a_{32}^2 \\
(1/2)(K_1 - K_2) \sin 2\phi = K_I a_{11}a_{12} + K_{II} a_{21}a_{22} + K_{III} a_{31}a_{32}
\]

- The coefficients consist of products of the rotation matrix

\[
\begin{pmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{pmatrix}
= \\
\begin{pmatrix}
\cos \gamma \cos \alpha - \cos \beta \sin \alpha \sin \gamma & \cos \gamma \sin \alpha + \cos \beta \cos \alpha \sin \gamma & \sin \gamma \sin \beta \\
- \sin \gamma \cos \alpha - \cos \beta \sin \alpha \cos \gamma & - \sin \gamma \sin \alpha + \cos \beta \cos \alpha \cos \gamma & \cos \gamma \sin \beta \\
\sin \beta \sin \alpha & - \sin \beta \cos \alpha & \cos \beta
\end{pmatrix}
\]
Numerical results for $K_I=1$, $K_{II}=2$, $K_{III}=3$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\varphi$</th>
<th>$K_1$</th>
<th>$K_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/6$</td>
<td>0</td>
<td>$\pi/6$</td>
<td>$\pi/3$</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>$\pi/6$</td>
<td>$\pi/6$</td>
<td>$\pi/6$</td>
<td>0.896</td>
<td>1.103</td>
<td>2.209</td>
</tr>
<tr>
<td>$\pi/6$</td>
<td>$\pi/3$</td>
<td>$\pi/6$</td>
<td>0.670</td>
<td>1.218</td>
<td>2.719</td>
</tr>
</tbody>
</table>

![Diagram of stress resultants](image)
Numerical results for $K_I = 3$, $K_{//} = 2$, $K_{///} = 1$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta$</th>
<th>$\gamma$</th>
<th>$\phi$</th>
<th>$K_1$</th>
<th>$K_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\pi/6$</td>
<td>0</td>
<td>$\pi/4$</td>
<td>$5\pi/12$</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>$\pi/6$</td>
<td>$\pi/8$</td>
<td>$\pi/4$</td>
<td>1.192</td>
<td>2.865</td>
<td>1.916</td>
</tr>
<tr>
<td>$\pi/6$</td>
<td>$\pi/4$</td>
<td>$\pi/4$</td>
<td>0.902</td>
<td>2.640</td>
<td>1.610</td>
</tr>
<tr>
<td>$\pi/6$</td>
<td>$3\pi/8$</td>
<td>$\pi/4$</td>
<td>0.669</td>
<td>2.578</td>
<td>1.192</td>
</tr>
</tbody>
</table>

Increasing $\beta$ from zero, we see how the effective permeabilities $K_1$ and $K_2$ are influenced more and more from $K_{///}$. The values of these effective permeabilities are between the extremal 3D values.
Original motivation for this work

- We investigated drainage flow from a 2D anisotropic porous medium into a ditch. General 2D permeability matrix

\[
\begin{pmatrix}
K_{xx} & K_{xy} \\
K_{yx} & K_{yy}
\end{pmatrix}
\]

- Our question was: Will our 2D drainage theory for a uniformly draining ditch be valid for a class of porous media with 3D anisotropy? The answer is yes. It is valid for general 3D anisotropy, defining the effective 2D permeability matrix presented here
Summary and conclusions

- A 2D flow in a general 3D anisotropic porous medium does not involve all six independent components of the full permeability tensor (matrix). It involves five, and reduces them to three.

- Three independent components of an effective 2D permeability tensor are sufficient to describe the 2D flow in the plane. These three are expressed by the principal values $K_1$, $K_2$ and the tilt angle $\varphi$ of the principal axes.

- One disadvantage with this approach is that it overlooks the passive flow perpendicular to the $x,y$ plane of the 2D flow.

- Another disadvantage is that we cannot reconstruct the full permeability tensor. Even if we took the passive flow components into account, we only have 5 equations to determine the 6 independent tensor components. The single tensor component that we cannot find with 2D flow is $K_{zz}$. 
APPENDIX X: Mathematica program for calculating the effective permeability matrix

\[
K_{\text{eff}} = \begin{pmatrix}
K_1 \cos^2 \varphi + K_2 \sin^2 \varphi & (1/2)(K_1 - K_2) \sin 2\varphi \\
(1/2)(K_1 - K_2) \sin 2\varphi & K_1 \sin^2 \varphi + K_2 \cos^2 \varphi
\end{pmatrix}
\]

\begin{verbatim}
ln[1]:= a11 = Cos[\[Alpha]]Cos[\[Gamma]] - Sin[\[Alpha]]Sin[\[Gamma]]Cos[\[Beta]]
ln[2]:= a12 = Sin[\[Alpha]]Cos[\[Gamma]] + Cos[\[Alpha]]Sin[\[Gamma]]Cos[\[Beta]]
ln[3]:= a13 = Sin[\[Beta]]Sin[\[Gamma]]
ln[4]:= a21 = -Cos[\[Alpha]]Sin[\[Gamma]] - Sin[\[Alpha]]Cos[\[Gamma]]Cos[\[Beta]]
ln[5]:= a22 = -Sin[\[Alpha]]Sin[\[Gamma]] + Cos[\[Alpha]]Cos[\[Gamma]]Cos[\[Beta]]
ln[6]:= a23 = Sin[\[Beta]]Cos[\[Gamma]]
ln[7]:= a31 = Sin[\[Alpha]]Sin[\[Beta]]
ln[8]:= a32 = -Cos[\[Alpha]]Sin[\[Beta]]
ln[9]:= a33 = Cos[\[Beta]]
ln[10]:= f[k1_, k2_, \[Phi]_] := k1 Cos[\[Phi]]^2 + k2 Sin[\[Phi]]^2 - k1 a11^2 - k1 a21^2 - k1 a31^2
ln[11]:= g[k1_, k2_, \[Phi]_] := k1 Sin[\[Phi]]^2 + k2 Cos[\[Phi]]^2 - k1 a12^2 - k1 a22^2 - k1 a32^2
ln[12]:= h[k1_, k2_, \[Phi]_] := (1/2) (k1 - k2) Sin[2 \[Phi]] - k1 a11 a12 - k1 a12 a12 - k1 a22 a22 - k1 a31 a32
ln[13]:= ff := k1 a11^2 + k1 a12^2 + k1 a13^2
ln[14]:= gg := k1 a22^2 + k1 a22^2 + k1 a32^2
ln[15]:= hh := k1 a11 a12 + k1 a21 a22 + k1 a31 a32
ln[16]:= kl[\[Phi]_] := (ff + gg) / 2 + hh / Sin[2 \[Phi]]
ln[17]:= k2[\[Phi]_] := (ff + gg) / 2 - hh / Sin[2 \[Phi]]
ln[18]:= los[\[Phi]_] := k1[\[Alpha]]Cos[\[Phi]]^2 + k2[\[Alpha]]Sin[\[Phi]]^2 - ff
ln[19]:= k1 := 1
ln[20]:= k11 := 2
ln[21]:= k11 := 3
\end{verbatim}