Two-dimensional flow in a porous medium with general anisotropy

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Darcy's law for flow in an isotropic porous medium

$$0 = -\nabla p - \frac{\mu}{K}\vec{v}$$

p is pressure
μ is dynamic viscosity
K is permeability
v is velocity (vector)

Henry Darcy (1956) formulated this linear law where a driving force is balanced by a resistance.
Analogous linear constitutive laws are:

Ohm's law of electric conduction

Fourier's law of heat conduction

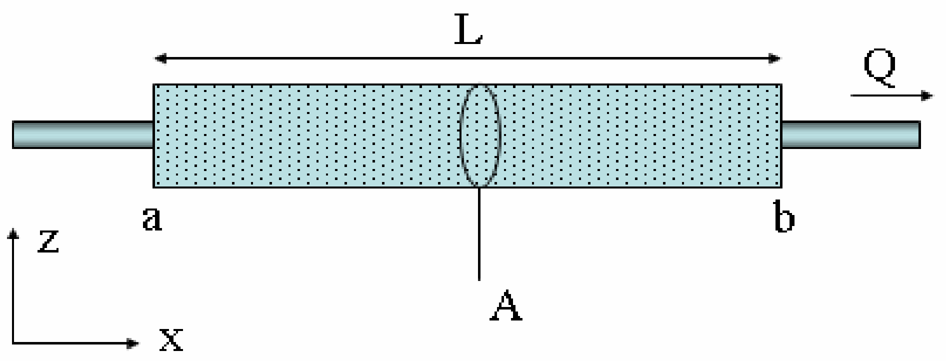
Fick's law of mass diffusion

Essentially these are first-order Taylor expansions

Darcy's law for flow in an isotropic porous medium: Newton's 2. law

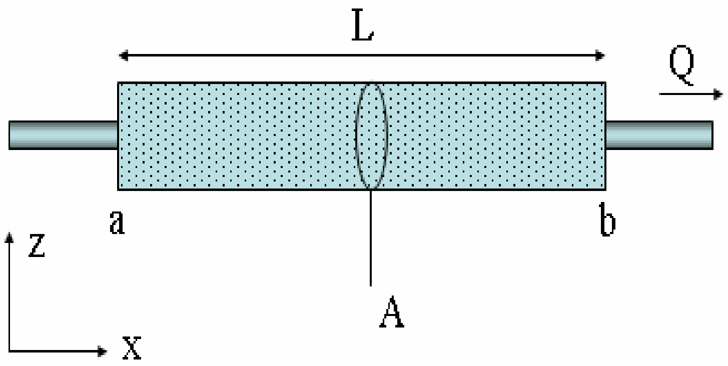
$$0 = -\nabla p - \frac{\mu}{K}\vec{v}$$

mass x acceleration = pressure force + resistance force (taken per volume, with acceleration term neglected)



Basic experiment for obtaining Darcy's law

Basic 1D experiment for obtaining Darcy's law



Tube (a)-(b) filled with a porous medium. Pressure fall $\Delta p = p_a - p_b$. Tube length *L*. Cross-section area *A*. Volume flux *Q.*

$$Q = K\frac{A}{\mu}\frac{p_a - p_b}{L} = -K\frac{A}{\mu}\frac{\partial p}{\partial x}$$

This is the observed relationship between pressure fall and volume flux. The permeability *K* is introduced as a proportionality constant.

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Darcy's law for flow in an isotropic porous medium

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Follows by generalizing to 3D the empirical relationship

$$Q = K\frac{A}{\mu}\frac{p_a - p_b}{L} = -K\frac{A}{\mu}\frac{\partial p}{\partial x}$$

by introducing as velocity the specific flux $v_x = Q/A$. Alternatively one may work with the average pore velocity nQ/A (n is porosity).

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Darcy's law for 1D flow in an anisotropic porous medium

The 1D version of Darcy's law

$$Q = K\frac{A}{\mu}\frac{p_a - p_b}{L} = -K\frac{A}{\mu}\frac{\partial p}{\partial x}$$

is rewritten for an anisotropic medium as

$$v_x = \frac{Q_x}{A} = -\frac{K_{xx}}{\mu}\frac{\partial p}{\partial x}$$

where K_{xx} is the permeability for flow in the x direction as driven by a gradient in the x direction

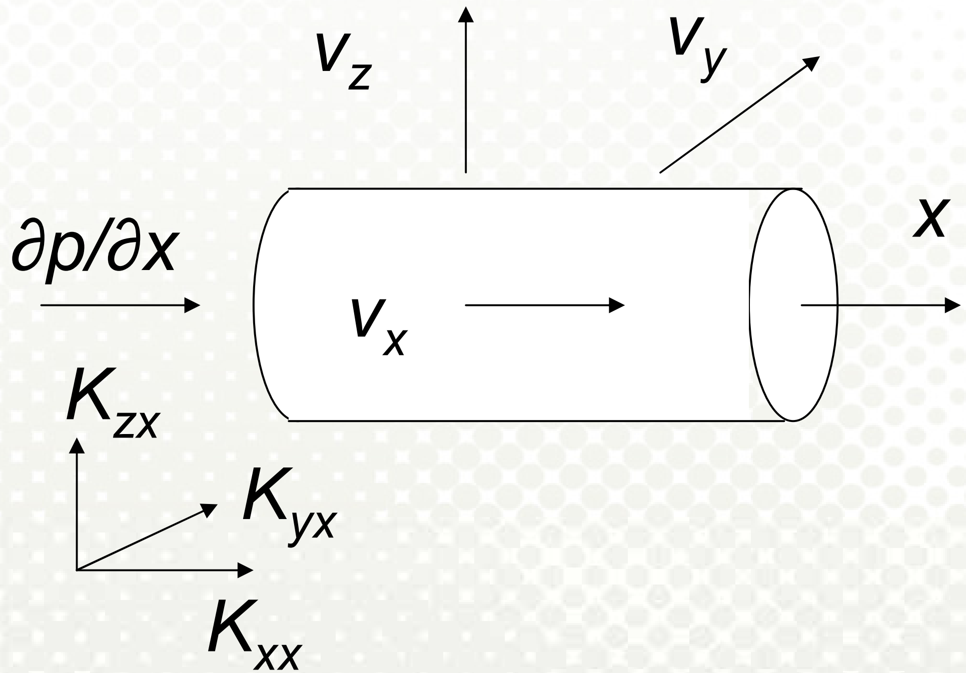
Darcy's law for 1D flow in an anisotropic porous medium

Darcy's law for an anisotropic porous medium with 1D pressure gradient $\partial p/\partial x=\Delta p/L_x$

$$v_x = \frac{Q_x}{A_x} = -\frac{K_{xx}}{\mu}\frac{\partial p}{\partial x}$$

$$v_y = \frac{Q_y}{A_y} = -\frac{K_{yx}}{\mu}\frac{\partial p}{\partial x}$$

$$v_z = \frac{Q_z}{A_z} = -\frac{K_{zx}}{\mu}\frac{\partial p}{\partial x}$$



K_{yx}, K_{zx} are the permeabilities for flows in the *y* and *z* directions as driven by a gradient in the *x* direction. Given by $K_{yx}/K_{xx}=v_y/v_x$, $K_{zx}/K_{xx}=v_z/v_x$

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Darcy's law for 3D flow in an anisotropic porous medium

Darcy's law for an anisotropic porous medium with 3D pressure gradient (*∂p/∂x, ∂p/∂y, ∂p/∂z)*

$$\mu\begin{pmatrix} v_x & v_y & v_z \end{pmatrix} = -\begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix}\begin{pmatrix} \partial p/\partial x \\ \partial p/\partial y \\ \partial p/\partial z \end{pmatrix}$$

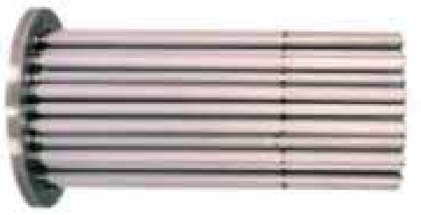
or, in index form

$$\mu v_i = K_{ij}\frac{\partial p}{\partial x_j}$$

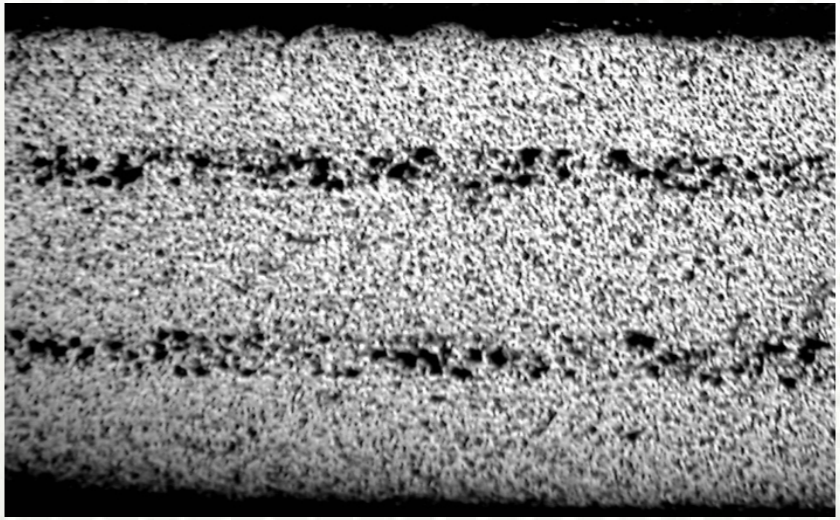
K_{ij} *(i=1,2,3) (j=1,2,3)* are the permeabilities for flow in the $x_1=x$, $x_2=y$ and $x_3=z$ directions as driven by gradients in the x_j *(j=1,2,3)* directions

3D anisotropic porous media

- We consider a homogeneous porous medium with general 3D anisotropy
- The anisotropy can be caused by fibres with preferred directions



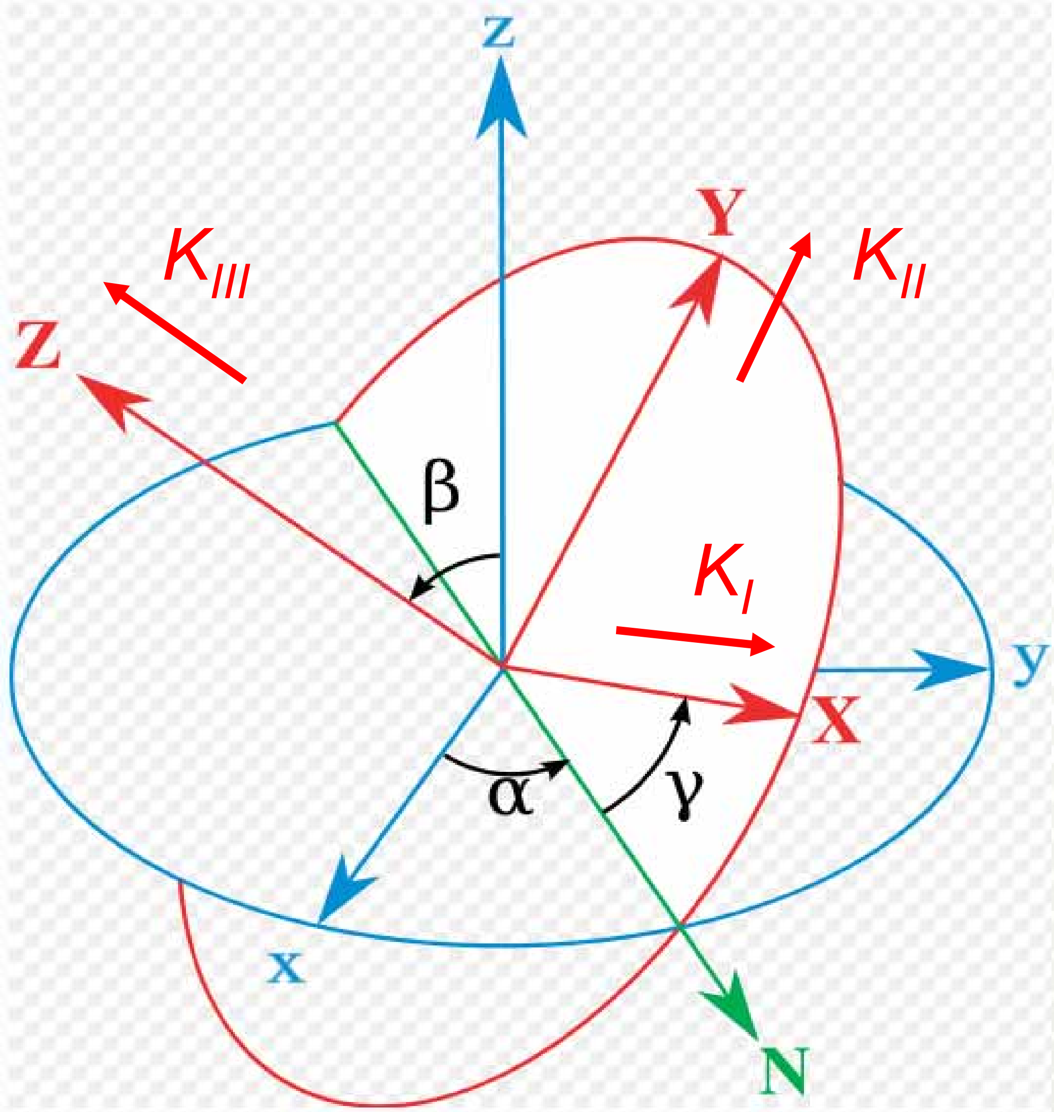
- The anisotropy can be caused by periodic layering



- These cases may have transverse isotropy: permeability along across layer. permeability in perpendicular direction
- Our analysis will assume general anisotropy, with arbitrary directions of principal axes

Flow in a 3D anisotropic porous medium

Coordinate axes *x,y,z*. Principal axes of permea-bility *X, Y, Z*, with permeabilities K_I, K_{II}, K_{III}. The two coordinate systems are linked by the three Euler angles *α, β, γ*



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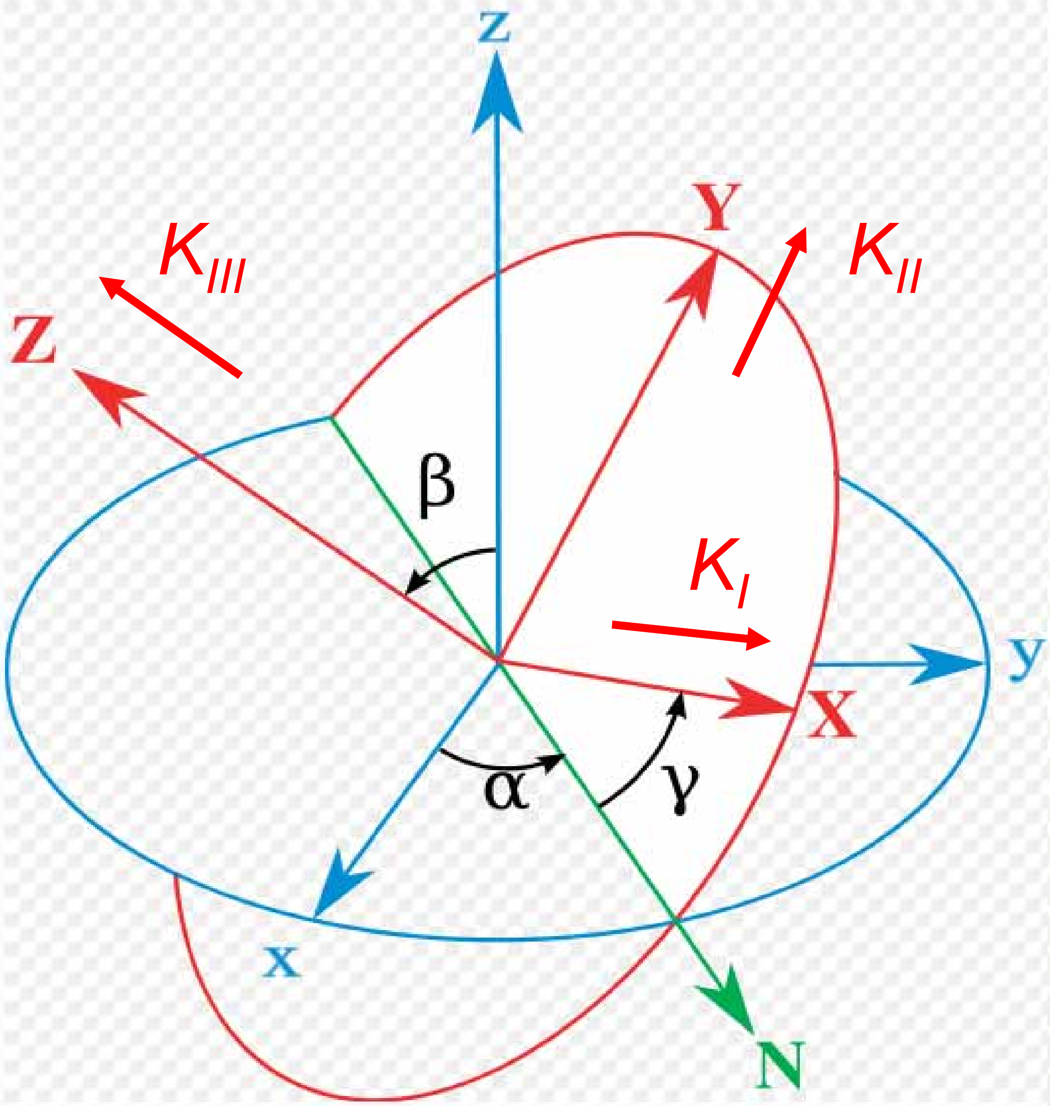
Flow in a 3D anisotropic porous medium

Principal axes of permeability are *X, Y, Z*. The permeability matrix is

$$\begin{pmatrix} K_I & 0 & 0 \\ 0 & K_{II} & 0 \\ 0 & 0 & K_{III} \end{pmatrix}$$

in the *X, Y, Z* system. The matrix is transformed to the *x,y,z* system



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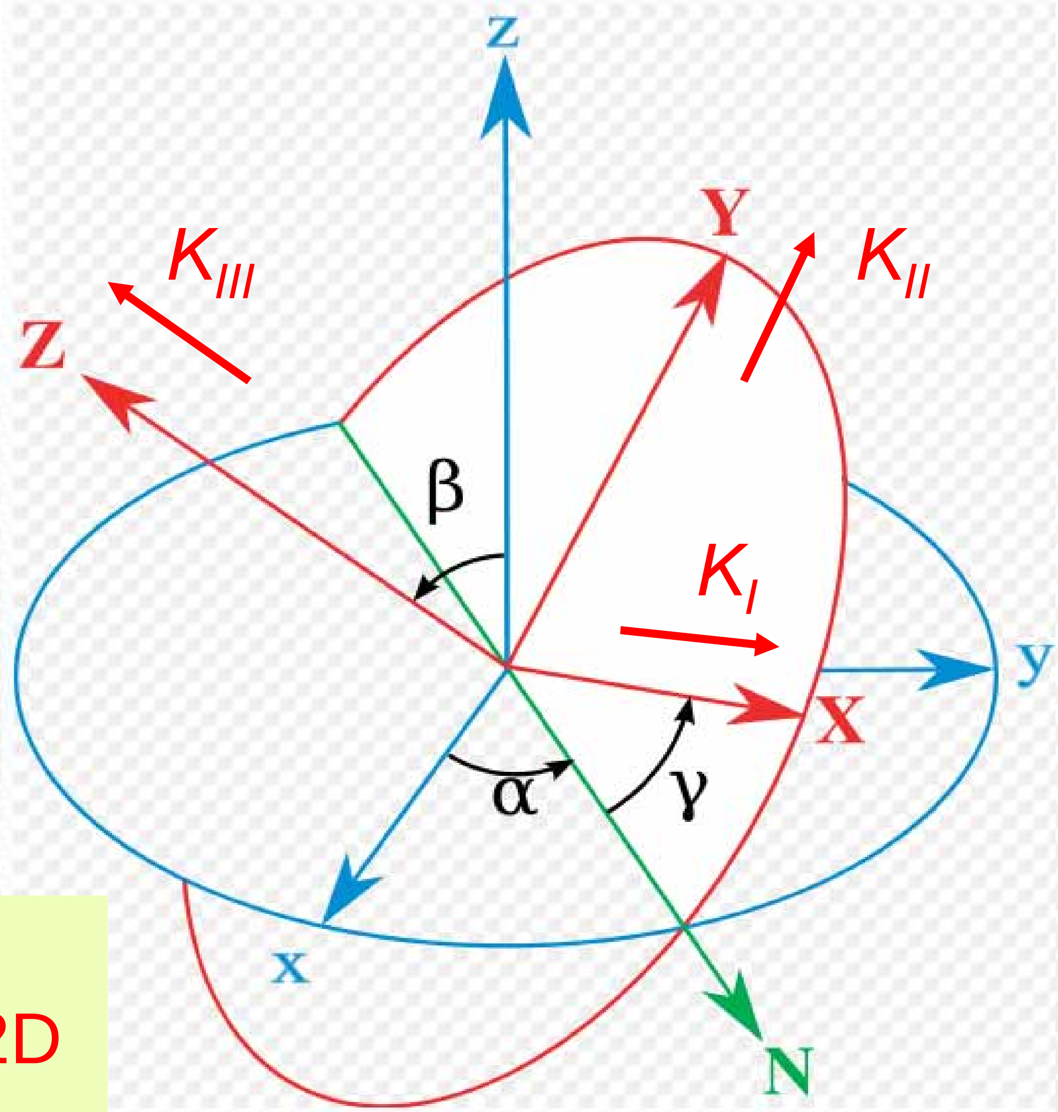
2D flow in a 3D anisotropic porous medium

The permeability matrix in the *x,y,z* system

$$\begin{pmatrix} K_{11} & K_{12} & K_{13} \\ K_{21} & K_{22} & K_{23} \\ K_{31} & K_{32} & K_{33} \end{pmatrix}$$

is given by the three principal permeabilites and the Euler angles (formulas omitted)



Now we restrict our attention to general 2D flow ($\partial/\partial z=0$)

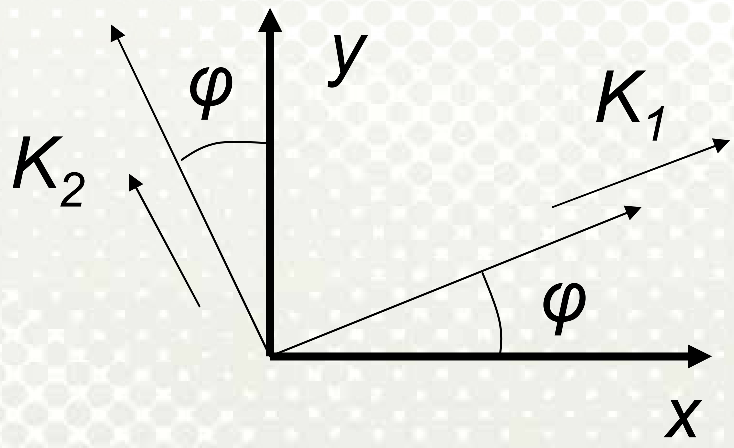
2D flow in a 3D anisotropic porous medium

- We introduce the pressure gradient that drives a 2D flow $(G_x, G_y) = -(\partial p/\partial x, \partial p/\partial y)$
- Note that this flow has a passive component in the *z* direction, even though there is no pressure gradient in the *z* direction
- Darcy's law for the 3D porous medium is written as

$$\vec{v} = \underline{K} \cdot \vec{G} = \vec{G} \cdot \underline{K}$$

- Here *K* is the 3D permeability tensor (matrix), which is symmetric. The velocity vector *v* has three components (v_x, v_y, v_z)

The idea is now to introduce an effective 2D description for the flow in the *(x,y)* plane. K_1 and K_2 are defined as effective 2D principal permeabilities, with axes rotated an angle φ compared with the coordinate axes *x* and *y*



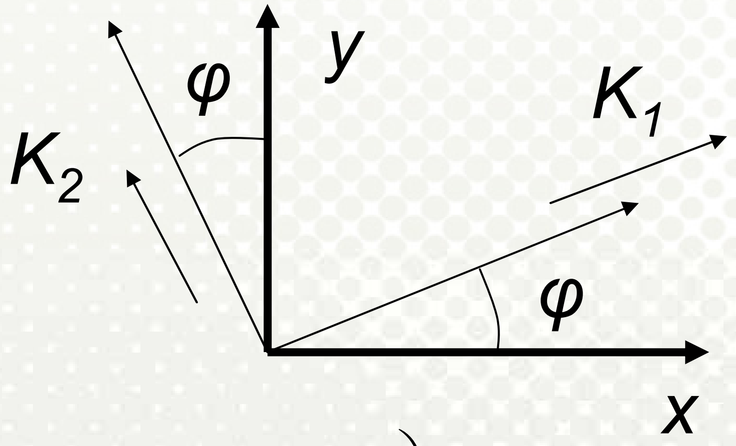
2D flow in a 3D anisotropic porous medium

- Darcy's law for 2D flow in a 3D porous medium with $(G_x, G_y) = (-\partial p/\partial x, -\partial p/\partial y)$
- $$\vec{v} = \underline{K} \cdot \vec{G} = \vec{G} \cdot \underline{K}$$
- The velocity vector *v* has three components (v_x, v_y, v_z) , but we are only interested in its 2D projection (v_x, v_y), which is given by the 2D formula

$$(v_x, v_y) = \vec{G} \cdot \underline{K}_{eff}$$

$$\underline{K}_{eff} = \begin{pmatrix} K_1 \cos^2\varphi + K_2 \sin^2\varphi & (1/2)(K_1 - K_2)\sin 2\varphi \\ (1/2)(K_1 - K_2)\sin 2\varphi & K_1 \sin^2\varphi + K_2 \cos^2\varphi \end{pmatrix}$$

An effective 2D description for the flow in the *(x,y)* plane takes K_1 and K_2 as effective 2D principal permeabilities, with axes rotated an angle φ



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2D flow in a 3D anisotropic porous medium

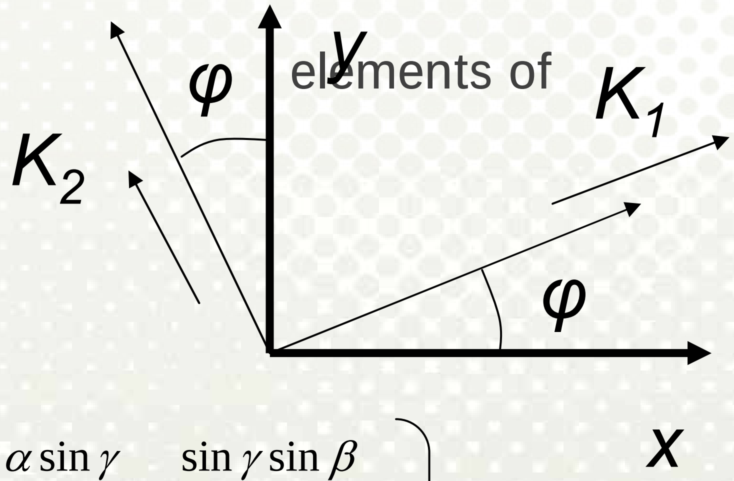
- The equations governing K_1, K_2 and φ are shown here. They can easily be solved numerically.

$$K_1 \cos^2\varphi + K_2 \sin^2\varphi = K_I a_{11}{}^2 + K_{II} a_{21}{}^2 + K_{III} a_{31}{}^2$$

$$K_1 \sin^2\varphi + K_2 \cos^2\varphi = K_I a_{12}{}^2 + K_{II} a_{22}{}^2 + K_{III} a_{32}{}^2$$

$$(1/2)(K_1 - K_2)\sin 2\varphi = K_I a_{11} a_{12} + K_{II} a_{21} a_{22} + K_{III} a_{31} a_{32}$$

- The coefficients consist of products of elements of the rotation matrix

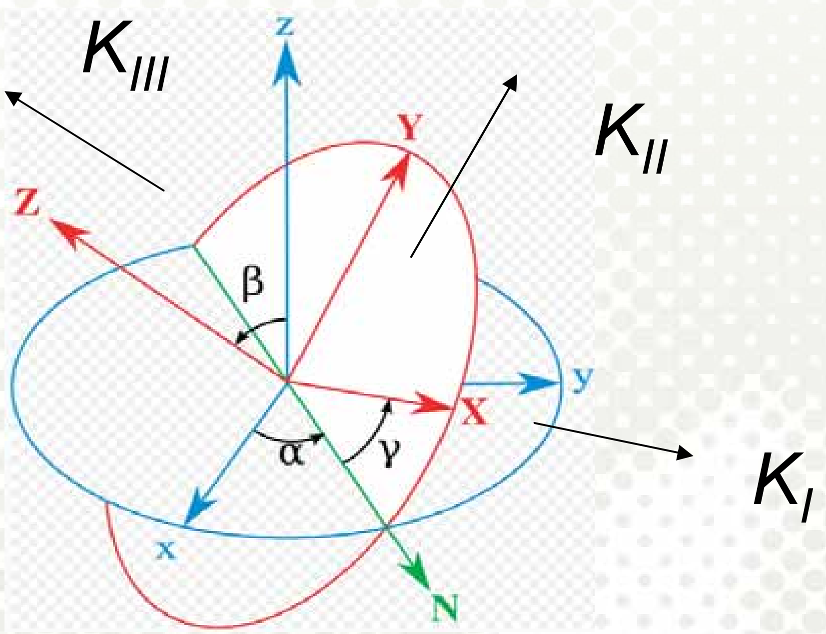


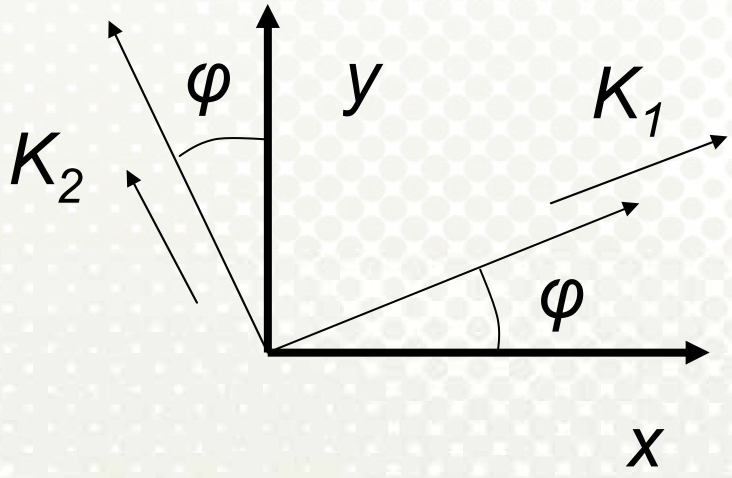
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} =$$

$$\begin{pmatrix} \cos\gamma\cos\alpha - \cos\beta\sin\alpha\sin\gamma & \cos\gamma\sin\alpha + \cos\beta\cos\alpha\sin\gamma & \sin\gamma\sin\beta \\ -\sin\gamma\cos\alpha - \cos\beta\sin\alpha\cos\gamma & -\sin\gamma\sin\alpha + \cos\beta\cos\alpha\cos\gamma & \cos\gamma\sin\beta \\ \sin\beta\sin\alpha & -\sin\beta\cos\alpha & \cos\beta \end{pmatrix}$$

Numerical results for $K_I=1, K_{II}=2, K_{III}=3$

α	β	γ	φ	K_1	K_2
$\pi/6$	0	$\pi/6$	$\pi/3$	1	2
$\pi/6$	$\pi/6$	$\pi/6$	.896	1.103	2.209
$\pi/6$	$\pi/3$	$\pi/6$	.670	1.218	2.719





Numerical results for K_I=3, K_{II}=2, K_{III}=1

α	β	γ	φ	K_1	K_2
$\pi/6$	0	$\pi/4$	$5\pi/12$	3	2
$\pi/6$	$\pi/8$	$\pi/4$	1.192	2.865	1.916
$\pi/6$	$\pi/4$	$\pi/4$	0.902	2.640	1.610
$\pi/6$	$3\pi/8$	$\pi/4$	0.669	2.578	1.192

Increasing β from zero, we see how the effective permeabilities K_1 and K_2 are influenced more and more from K_{III}. The values of these effective permeabilities are between the extremal 3D values

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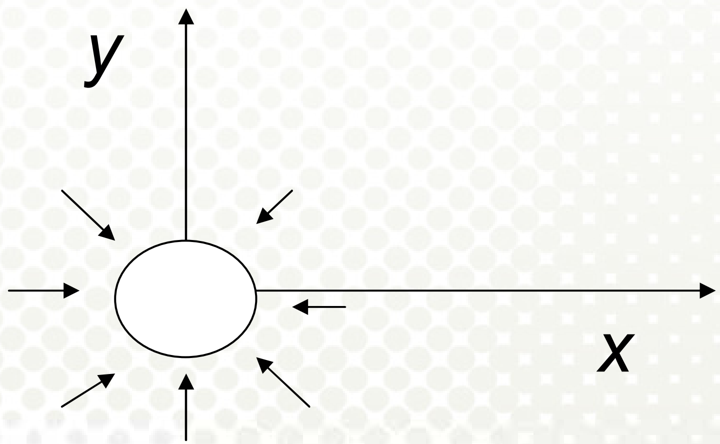
Original motivation for this work

- We investigated drainage flow from a 2D anisotropic porous medium into a ditch. General 2D permeability matrix

$$\begin{pmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{pmatrix}$$



- Our question was: Will our 2D drainage theory for a uniformly draining ditch be valid for a class of porous media with 3D anisotropy? The answer is yes. It is valid for general 3D anisotropy, defining the effective 2D permeability matrix presented here

Summary and conclusions

- A 2D flow in a general 3D anisotropic porous medium does not involve all six independent components of the full permeability tensor (matrix). It involves five, and reduces them to three.
- Three independent components of an effective 2D permeability tensor are sufficient to describe the 2D flow in the plane. These three are expressed by the principal values K_1, K_2 and the tilt angle φ of the principal axes.
- One disadvantage with this approach is that it overlooks the passive flow perpendicular to the x,y plane of the 2D flow
- Another disadvantage is that we cannot reconstruct the full permeability tensor. Even if we took the passive flow components into account, we only have 5 equations to determine the 6 independent tensor components. The single tensor component that we cannot find with 2D flow is K_{zz}

APPENDIX: Mathematica program for calculating the effective permeability matrix

Mathematica 5.2 - [dalen.nb *]

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dalen.nb *

```
In[1]:= a11 = Cos[α] Cos[γ] - Sin[α] Sin[γ] Cos[β]
In[2]:= a12 = Sin[α] Cos[γ] + Cos[α] Sin[γ] Cos[β]
In[3]:= a13 = Sin[β] Sin[γ]
In[4]:= a21 = -Cos[α] Sin[γ] - Sin[α] Cos[γ] Cos[β]
In[5]:= a22 = -Sin[α] Sin[γ] + Cos[α] Cos[γ] Cos[β]
In[6]:= a23 = Sin[β] Cos[γ]
In[7]:= a31 = Sin[α] Sin[β]
In[8]:= a32 = -Cos[α] Sin[β]
In[9]:= a33 = Cos[β]
In[10]:= f[k1_, k2_, φ_] := k1 Cos[φ] ^ 2 + k2 Sin[φ] ^ 2 - kI a11 ^ 2 - kII a21 ^ 2 - kIII a31 ^ 2
In[11]:= g[k1_, k2_, φ_] := k1 Sin[φ] ^ 2 + k2 Cos[φ] ^ 2 - kI a12 ^ 2 - kII a22 ^ 2 - kIII a32 ^ 2
In[12]:= h[k1_, k2_, φ_] := (1 / 2) (k1 - k2) Sin[2 φ] - kI a11 a12 - kII a21 a22 - kIII a31 a32
In[13]:= ff = kI a11 ^ 2 + kII a21 ^ 2 + kIII a31 ^ 2
In[14]:= gg = kI a12 ^ 2 + kII a22 ^ 2 + kIII a32 ^ 2
In[15]:= hh = kI a11 a12 + kII a21 a22 + kIII a31 a32
In[16]:= k1[φ_] := (ff + gg) / 2 + hh / Sin[2 φ]
In[17]:= k2[φ_] := (ff + gg) / 2 - hh / Sin[2 φ]
In[18]:= los[φ_] := k1[α] Cos[φ] ^ 2 + k2[α] Sin[φ] ^ 2 - ff
In[29]:= kI = 1
In[30]:= kII = 2
In[31]:= kIII = 3
```

$$\underline{K}_{eff} = \begin{pmatrix} K_1 \cos^2\varphi + K_2 \sin^2\varphi & (1/2)(K_1 - K_2)\sin 2\varphi \\ (1/2)(K_1 - K_2)\sin 2\varphi & K_1 \sin^2\varphi + K_2 \cos^2\varphi \end{pmatrix}$$